Nesbitt's inequality (England 1903)

$$\vdash \quad \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ge\frac{3}{2} \qquad \text{where} \quad a, b, c > 0$$

A.M. ≥ G. M. ≥ H.M. (Arithmetic – Geometric – Harmonic mean inequality)

In ***Proof 1*** and ***Proof 2*** below we use the following:

$$\frac{x+y+z}{3}\ge\sqrt[3]{xyz}\ge\frac{3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}} \qquad \text{where} \quad x, y, z > 0.$$

Proof 1

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ge\frac{3}{2}$$

$$\Leftrightarrow \quad \left(1+\frac{a}{b+c}\right)+\left(1+\frac{b}{c+a}\right)+\left(1+\frac{c}{a+b}\right)\ge 3+\frac{3}{2}$$

$$\Leftrightarrow \quad \frac{a+b+c}{b+c}+\frac{a+b+c}{c+a}+\frac{a+b+c}{a+b}\ge\frac{9}{2}$$

$$\Leftrightarrow \quad 2(a+b+c)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)\ge 9$$

$$\Leftrightarrow \quad [(a+b)+(b+c)+(c+a)]\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)\ge 9$$

$$\Leftrightarrow \quad \frac{(a+b)+(b+c)+(c+a)}{3}\ge\frac{3}{\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)}$$

$\Leftrightarrow$ A.M. ≥ H.M. for 3 variables

Proof 2

Apply A.M. ≥ G.M. for two sets of data and find the product:

$$\frac{(a+b)+(b+c)+(c+a)}{3}\,\frac{\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}}{3}\ge\sqrt[3]{(a+b)(b+c)(c+a)}\sqrt[3]{\left(\frac{1}{b+c}\right)\left(\frac{1}{c+a}\right)\left(\frac{1}{a+b}\right)}$$

$$\Rightarrow \quad \frac{2(a+b+c)}{3}\,\frac{\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}}{3}\ge\sqrt[3]{(a+b)(b+c)(c+a)\left(\frac{1}{b+c}\right)\left(\frac{1}{c+a}\right)\left(\frac{1}{a+b}\right)}$$

$$\Rightarrow \quad \frac{2}{9}\,\frac{a+b+c}{b+c}+\frac{a+b+c}{c+a}+\frac{a+b+c}{a+b}\ge 1$$

$$\Rightarrow \quad \left(1+\frac{a}{b+c}\right)+\left(1+\frac{b}{c+a}\right)+\left(1+\frac{c}{a+b}\right)\ge\frac{9}{2}$$

$$\Rightarrow \quad \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ge\frac{3}{2}$$

Proof 3

A.M. ≥ G.M.

$$\frac{a+b}{b+c}+\frac{a+c}{b+c}+\frac{b+c}{a+c}+\frac{a+b}{a+c}+\frac{a+c}{a+b}+\frac{b+c}{a+b}\geq 6\sqrt[6]{\left(\frac{a+b}{b+c}\right)\left(\frac{a+c}{b+c}\right)\left(\frac{b+c}{a+c}\right)\left(\frac{a+b}{a+c}\right)\left(\frac{a+c}{a+b}\right)\left(\frac{b+c}{a+b}\right)}$$

$$\frac{a+b}{b+c}+\frac{a+c}{b+c}+\frac{b+c}{a+c}+\frac{a+b}{a+c}+\frac{a+c}{a+b}+\frac{b+c}{a+b}\geq 6$$

$$\frac{2a}{b+c}+\frac{2b}{a+c}+\frac{2c}{a+b}+3\geq 6$$

$$\therefore \quad \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq \frac{3}{2}$$

CBS inequality (Cauchy – Bunyakovskii – Schwarz inequality)

In ***Proof 4*** below we use:

If $x_1, x_2, x_3 ; y_1, y_2, y_3 \in \mathbf{R}$,

then $\left(x_1^2+x_2^2+x_3^2\right)\left(y_1^2+y_2^2+y_3^2\right)\geq (x_1y_1+x_2y_2+x_3y_3)^2$

Proof 4

Put $x_1=\sqrt{a+b},\ x_2=\sqrt{b+c},\ x_3=\sqrt{c+a};\quad y_1=\sqrt{\frac{1}{a+b}},\ y_2=\sqrt{\frac{1}{b+c}},\ y_3=\sqrt{\frac{1}{c+a}}$

in CBS inequality above.

$$[(a+b)+(b+c)+(c+a)]\left[\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right]\geq (1+1+1)^2$$

$$\Rightarrow \quad 2(a+b+c)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)\geq 9$$

$$\Rightarrow \quad \frac{a+b+c}{b+c}+\frac{a+b+c}{c+a}+\frac{a+b+c}{a+b}\geq \frac{9}{2}$$

$$\Rightarrow \quad \left(1+\frac{a}{b+c}\right)+\left(1+\frac{b}{c+a}\right)+\left(1+\frac{c}{a+b}\right)\geq \frac{9}{2}$$

$$\Rightarrow \quad \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq \frac{3}{2}$$

Rearrangement inequality

We introduce the scalar product $\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = x_1y_1 + x_2y_2 + x_3y_3$

If $x_1 \le x_2 \le x_3$; $y_1 \le y_2 \le y_3$, $x_i, y_i \in \mathbf{R}$,

then $\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \ge \begin{bmatrix} x_1 & x_2 & x_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \ge \begin{bmatrix} x_1 & x_2 & x_3 \\ y_3 & y_2 & y_1 \end{bmatrix}$

where z_1, z_2, z_3 is any permutation of y_1, y_2, y_3.

that is the scalar product is maximal if the two sequences are sorted in the same way and is minimal if sorted oppositely.

Proof 5

The Nesbitt's inequality can be proved by assuming without lost of generality:

$a \le b \le c$, we have $b + c \ge c + a \ge a + b$

then $\frac{1}{b+c} \le \frac{1}{c+a} \le \frac{1}{a+b}$

By Rearrangement inequality,

(1) $\begin{bmatrix} a & b & c \\ \frac{1}{b+c} & \frac{1}{c+a} & \frac{1}{a+b} \end{bmatrix} \ge \begin{bmatrix} a & b & c \\ \frac{1}{c+a} & \frac{1}{a+b} & \frac{1}{b+c} \end{bmatrix}$

(2) $\begin{bmatrix} a & b & c \\ \frac{1}{b+c} & \frac{1}{c+a} & \frac{1}{a+b} \end{bmatrix} \ge \begin{bmatrix} a & b & c \\ \frac{1}{a+b} & \frac{1}{b+c} & \frac{1}{c+a} \end{bmatrix}$

Adding (1) and (2), $2\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right) \ge 3$

$$\therefore \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2}$$

Normalization

Since the left hand side of the Nesbitt's inequality is cyclic in a, b, c, we can employ the method of normalization, that is, we assume that $a + b + c = 1$.

Proof 6

Since a + b + c = 1, we observe that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=\frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c}=\frac{1}{1-a}-1+\frac{1}{1-b}-1+\frac{1}{1-c}-1$$

Nesbitt's inequality is reduced to an easier form : $\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}\ge\frac{9}{2}$

To prove this, we apply A.M. ≥ H.M., $\frac{(1-a)+(1-b)+(1-c)}{3}\ge\frac{3}{\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}}$

Rearranging, we get,

(3) $$\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}\ge\frac{9}{3-(a+b+c)}=\frac{9}{3-1}=\frac{9}{2}$$

To get back the Nesbitt's inequalities, observe that $\frac{a}{a+b+c}+\frac{b}{a+b+c}+\frac{c}{a+b+c}=1$,

so we replace in (3), a by $\frac{a}{a+b+c}$, b by $\frac{b}{a+b+c}$ and c by $\frac{c}{a+b+c}$.

Note

If a, b, c are sides of a triangle we also have: $\frac{3}{2}\le\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}<2$

The right hand side inequality can be proved by **Triangular inequality**: a + b > c

We easily have : b + c , c + a , a + b $>\frac{1}{2}(a+b+c)$

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}<\frac{a}{\frac{1}{2}(a+b+c)}+\frac{b}{\frac{1}{2}(a+b+c)}+\frac{c}{\frac{1}{2}(a+b+c)}=2$$

Exercise

Proof 7

(a) By considering $\left(\frac{a}{b+c}-\frac{1}{2}\right)^2$, show that $\frac{a}{b+c}\ge\frac{\frac{8a}{b+c}-1}{\frac{a}{b+c}+1}=\frac{8a-b-c}{4(a+b+c)}$, where $a,b,c>0$.

(b) Hence prove the Nesbitt's inequality:

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ge\frac{3}{2}\quad\text{where } a, b, c>0 .$$